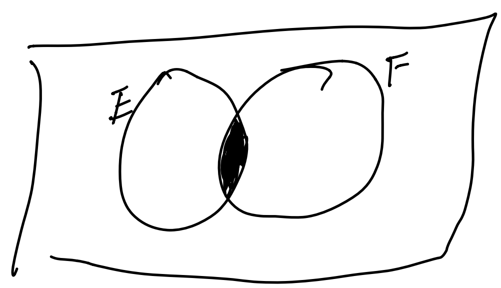
Lecture #4

Axiomatic Probability Continued

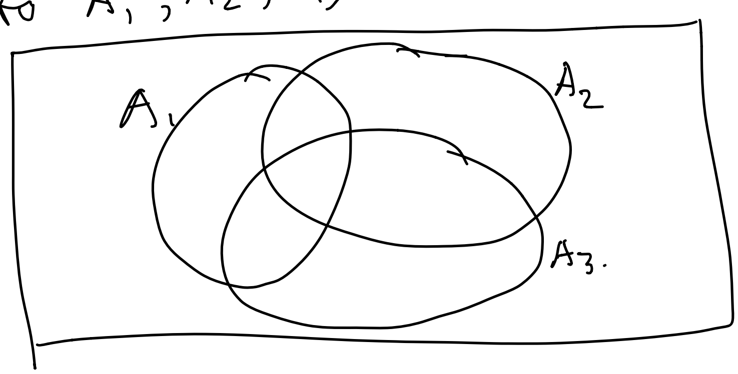
Last time, we showed that given any two events, E and F, we have that

$$P(E \cup F) = P(E) + P(F) - P(EF).$$

Intuitively this is b because we have double counted the intersection:



Let's think about what happens with three sets. We want to know the probability $P(A \cup B \cup C)$ for events A_1, A_2, A_3



In the diagram, we identify the probability of each event with its area.

First approximation:

$$P(A_1 \cup A_2 \cup A_3) \approx P(A_1) + P(A_2) + P(A_3). \quad (1)$$

What's the problem here? For each pair $A_i\ A_j$, $i \neq j$, we have the region $A_i \cap A_j$ and

$$A_i \cap A_j \subseteq A_i$$
$$\subseteq A_j$$

So when we include the area $P(A_i)$ and $P(A_j)$, we are including $P(A_i \cap A_j)$ twice; once as a sub-area of $P(A_i)$ and once as a sub area of $P(A_j)$. Therefore, for our second order approximation we need to subtract one of these double counted regions <u>per pair</u>:

$$P(A_1 \cup A_2 \cup A_3) \approx P(A_1) + P(A_2) + P(A_3) - P(A_1 \cap A_2) - P(A_2 \cap A_3) - P(A_1 \cap A_3). \quad (2)$$

Finally, we need to address the region

$$A_1 \cap A_2 \cap A_3 \subseteq A_1$$
$$\subseteq A_2$$
$$\subseteq A_3.$$

Since this region is contained in all three events, our original approximation counted this area <u>three</u> times. However, we also have

$$A_1 \cap A_2 \cap A_3 \subseteq A_1 \cap A_2$$
$$\subseteq A_2 \cap A_3$$
$$\subseteq A_3 \cap A_4,$$

($A_i A_j = A_i \cap A_j$; This is just notation)

and so we have subtracted this area off all together in our second approximation! Thus, for our last approximation, we have

$$P(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2) + P(A_3) - P(A_1 A_2) - P(A_2 A_3) - P(A_1 A_3) + P(A_1 A_2 A_3).$$

This technique, in general, is called the "inclusion-exclusion principle":

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{r=1}^{n} (-1)^{r+1} \sum_{i_1 < \cdots < i_r} P(A_1 \cdots A_r)$$

2.5 Sample Spaces with equally likely outcomes:

In many real life situations we consider the situation where we have a sample space with equally likely outcomes; that is, our space is a finite set $S = \{1, \ldots, N\}$, and

$$P(\{i\}) = 1/N$$

for each $i \in S$.

Examples: 1) Flipping a coin
2) Rolling a die
3) Choosing a card from a shuffled deck.

Ex: In a 5-card poker hand, a royal flush consists of 10, J, Q K, A, all of the same suit. Assuming each hand is equally likely, what is the probability of being dealt a royal flush?

↳ There are $\binom{52}{5}$ possible hands. Since they must all be of the same suit, there are only 4 possibilities. Hence the probability is $4/\binom{52}{5} \approx 0.0000154$
i.e. 0.000154% chance.

Ex: In a 5-card poker game, a straight flush consists of 5 cards of the same suit in sequential order, but not including ~~the~~ royal flushes; i.e, the highest straight flush is 9, 10, J, Q, K of a given suit, and the lowest is A, 2, 3, 4, 5.
Therefore for each suit, the straight flushes are determined by the lowest card in the

hand, in the range A, 2, 3, 4, 5, 6, 7, 8, 9.

Hence for each suit, there are 9 possible straight flushes. Since there are 4 suits, there are $9 \cdot 4 = 36$ possible straight flushes, giving them a probability of

$$36 / \binom{52}{5} \approx 0.000013$$

i.e. 0.0013% chance.

Ex A flush (in 5-card poker) is a hand where all cards are of the same suit.

For each suit, there are 13 cards and so there are $\binom{13}{5}$ possible flushes per suit. Since there are 4 suits, there are $4\binom{13}{5}$ possible flushes and hence a probability of

$$\frac{4\binom{13}{5}}{\binom{52}{5}} \approx 0.00198$$

i.e. 0.198%.

Ex: In 5-card poker, a full house is a hand with a pair of one denomination, and three of another, i.e. 2 fives and 3 queens, 2 kings 3 jacks, etc.

Let's count the number of ways we can get a full house.

First, we choose the pair: per suit, there are 13 different options, A, 2, 3, ... 10, J, Q, K, so we have 13 options for the face value.

Once we have the face value, we need to choose the pair. For example, if we are getting a pair of 3's, we can choose (hearts, spades), (hearts, clubs) (hearts, diamonds), (spades, clubs) ... etc, and so there are $13\binom{4}{2}$ ways to get the pair.

Now we want to get the <u>three of a kind</u>. Now there are only 12 options for the face value of the cards, since we already have a pair and there <u>is not</u> 5 suits. So there are $12\binom{4}{3}$ many options for three of a kind.

So all together:

$$\frac{12\binom{4}{2}\ 13\binom{4}{3}}{\binom{52}{5}} \simeq 0.014.$$

i.e. 1.4% chance.

<u>Read:</u> Example 5m from the text.